

# COMPACTIFYING NORMAL ALGEBRAIC SPACES

DAN EDIDIN

**ABSTRACT.** The author wrote this note after being asked about the existence of compactifications of algebraic spaces. Subsequent to posting the article to the math arXiv, the author learned from Yutakaa Matsuura that the results of this paper had been proved by Raoult in his 1971 paper [Rao] using the same techniques. Since Raoult's article may be unknown to those working in the field, the author is keeping this preprint on the arXiv server. However, he makes no claim of originality.

A classic theorem of Nagata [Nag1, Nag2] states that any variety may be embedded into a complete scheme. Nagata's proof was translated to the language of schemes by Deligne<sup>1</sup>. It is now known that any separated scheme of finite type over a quasi-compact and quasi-separated base scheme may be embedded as an open subscheme in a scheme which is proper over the base [Con, Theorem 4.1].

Nagata's completion result is an important technical tool in a number of contexts; for example it is used in the construction of higher direct images in étale cohomology with compact support ([Mil]). A natural problem is to determine whether separated algebraic spaces and, more generally, separated Deligne-Mumford stacks admit compactifications.

Unfortunately the essential idea used in all proofs of Nagata's theorem is not available in the category of algebraic spaces. The point is that any scheme admits a cover by Zariski open subschemes which are quasi-projective over the ground scheme. The open sets in the cover admit obvious compactifications and a global compactification may be constructed via a delicate gluing process. Because algebraic spaces and Deligne-Mumford stacks are only étale locally quasi-projective schemes, it is not apparent how to carry over the compactification strategy used for schemes.

The purpose of this note is to show that using Galois descent it is possible to give an easy proof that normal algebraic spaces admit compactifications.

---

<sup>1</sup>See [Con] or [Voj] for an exposition of Deligne's argument. An independent scheme-theoretic proof was given by Lütkebohmert [Lüt].

**Theorem 0.1.** *Let  $X$  be a normal algebraic space which is separated and of finite type over a Noetherian scheme  $S$ . Then there is an algebraic space  $\overline{X}$  which is proper over  $S$  and contains  $X$  as a dense open subspace.*

*Proof.* Since  $X$  is a normal algebraic space it is generically a normal scheme and hence has a function field  $K(X)$ . Also,  $X$  is Noetherian as it is of finite type over the Noetherian scheme  $S$ . By [LMB, Corollaire 16.6.2] the normal algebraic space  $X$  is a geometric quotient  $Y/\Gamma$  where  $Y$  is a normal scheme and  $\Gamma = \text{Gal}(K(Y)/K(X))$  is finite. By Nagata's theorem the scheme  $Y$  has a compactification  $\overline{Y}$  which is proper over  $S$  (and hence Noetherian). The following construction (which we learned from Sumihiro's paper [Sum]) allows us to replace  $\overline{Y}$  by a  $\Gamma$ -equivariant compactification.

Enumerate the elements of  $\Gamma$  as  $\gamma_1 = e, \dots, \gamma_n$  and define a pairing  $l: [1, n] \times [1, n] \rightarrow [1, n]$  by  $\gamma_i \gamma_j = \gamma_{l(i,j)}$ . Let  $\Gamma$  act on the  $n$ -fold product over  $S$ ,  $\overline{Y}^n$ , by the rule  $\gamma_i(y_1, \dots, y_n) = (y_{l(1,i)}, \dots, y_{l(n,i)})$ . With our chosen  $\Gamma$  action the embedding  $s: Y \rightarrow \overline{Y}^n$ ,  $z \mapsto (\gamma_1 z, \dots, \gamma_n z)$  is  $\Gamma$ -equivariant. Let  $\overline{W}$  be the closure of the scheme theoretic image of  $s$ . Since  $s$  is  $\Gamma$ -equivariant, there is an action of  $\Gamma$  on  $\overline{W}$  which extends the action on the dense open subscheme  $Y$ . Moreover,  $\overline{W}$  is proper over  $S$  (and hence Noetherian) since it is a closed subscheme of the proper  $S$ -scheme  $\overline{Y}^n$ .

Let  $\overline{X}$  be the geometric quotient of  $\overline{W}$  by the action of  $\Gamma$  (such quotients always exist in the category of algebraic spaces). Since  $\overline{W} \rightarrow \overline{X}$  is finite, the algebraic space  $\overline{X}$  is proper over  $S$  and contains the quotient  $X = Y/\Gamma$  as a dense open subspace.  $\square$

As a corollary we obtain the following compactification result for separated morphisms of algebraic spaces over a Noetherian base scheme.

**Corollary 0.2.** *Let  $X \xrightarrow{f} Y$  be a morphism of algebraic spaces which is separated and of finite type. Assume that  $Y$  (and hence  $X$ ) is separated and of finite type over a Noetherian base scheme  $S$  and that  $X$  is normal. Then there exists an algebraic space  $\overline{X}$  which is proper over  $Y$  and contains  $X$  as a dense open subspace.*

*Proof.* By Theorem 0.1 we know that the normal algebraic space  $X$  has an  $S$ -compactification  $\tilde{X}$ . The map  $\tilde{X} \times_S Y \rightarrow Y$  obtained by base change is proper and contains  $X \times_S Y$  as a dense open subspace. Let  $\overline{X}$  be the closure (in the sense of algebraic spaces) of the graph of  $f$  in  $\tilde{X} \times_S Y$ .  $\square$

**Acknowledgement:** The author thanks Martin Olsson for suggesting to him the problem of compactifying algebraic spaces.

## REFERENCES

- [Con] Brian Conrad, *Deligne's notes on Nagata compactifications*, preprint (2007).
- [LMB] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Springer-Verlag, Berlin, 2000.
- [Lüt] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), no. 1, 95–111.
- [Mil] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.
- [Nag1] Masayoshi Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10.
- [Nag2] ———, *A generalization of the imbedding problem of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **3** (1963), 89–102.
- [Rao] Jean-Claude Raoult, *Compactification des espaces algébriques normaux*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A766–A767.
- [Sum] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.
- [Voj] Paul Vojta, *Nagata's embedding theorem*, arXiv:math.AG/07061907 (2007).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211

*E-mail address:* edidin@math.missouri.edu